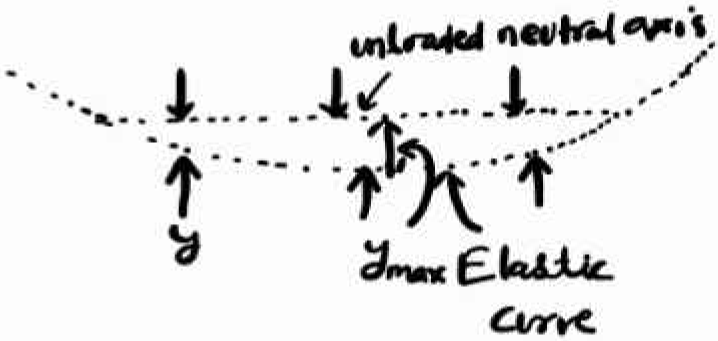
UNIT :5

- Slope and deflection: Relationship between moment, Slope and Deflection
- Determine slopes and deflections of the determinate beams (Simply supported and cantilever)
- Double integration method
- Macaulay's method
- Moment area method
- Conjugate beam method

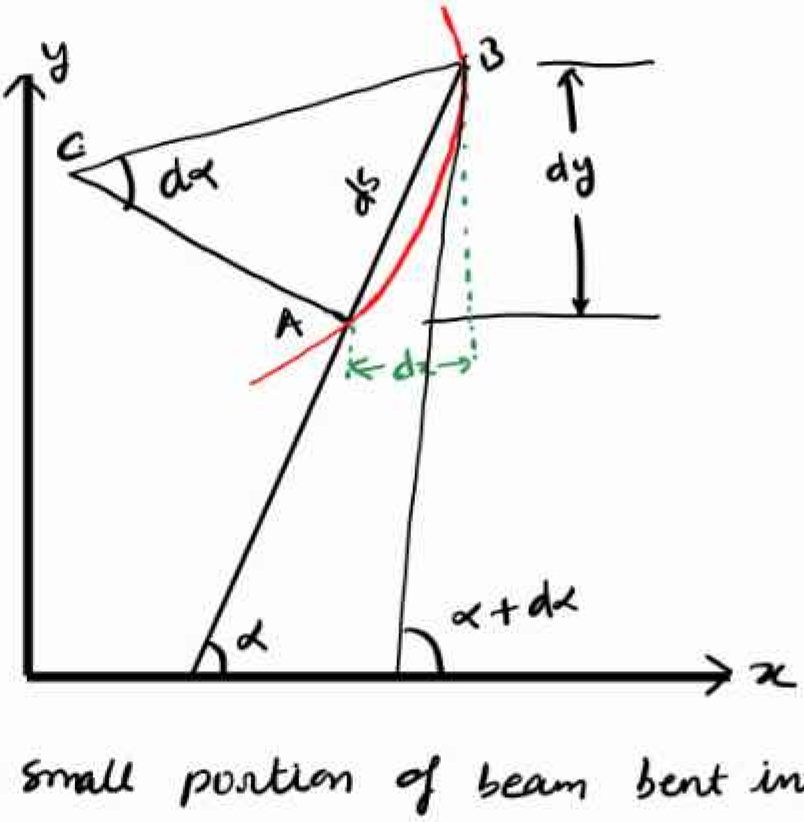
Beam without load

Beam with load



Under load neutral axis becomes a curved line and is called the elastic curve. The deflection 'y' is vertical distance between a point on the elastic curve and the unloaded neutral axis.

Relation between moment, slope, and deflection



In given figure, a small portion of beam bent into an arc.

ds = length of beam AB

C = Centre of the arc

α = Angle which the tangent at A makes with XX-axis, and

$\alpha + d\alpha$ = Angle which the tangent at B makes with XX axis.

From geometry,

$$\angle ACB = d\alpha \quad \text{and} \quad ds = R\,d\alpha$$

$$\therefore \quad R = \frac{ds}{d\alpha}$$

$$\frac{1}{R} = \frac{d\alpha}{ds} \qquad \text{——— (1)}$$

If the co-ordinate of point A are x and y, then

$$\tan\alpha = \frac{dy}{dx} \quad (\text{or}) \quad \alpha = \frac{dy}{dx} \qquad \text{——— (2)}$$

take $\tan\alpha = \alpha$ as α is very small

differentiating equation (2) w.r.t. x

$$\frac{d\alpha}{dx} = \frac{d^2y}{dx^2}$$

$$\frac{1}{R} = \frac{d^2y}{dx^2}$$

By using bending eqn.

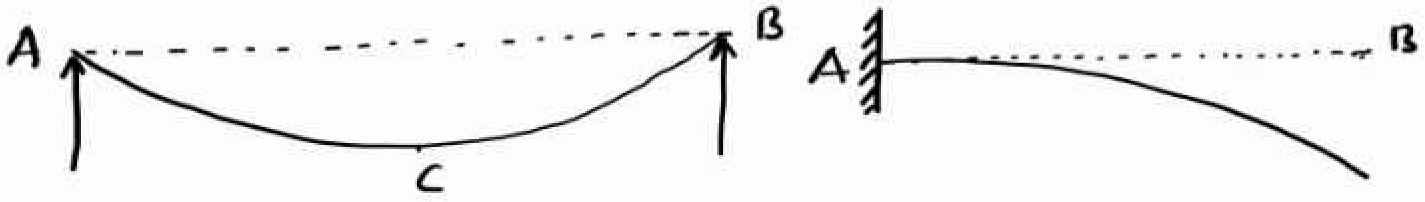
$$\frac{M}{I} = \frac{E}{R}$$

$$\frac{M}{I} = E \times \frac{d^2y}{dx^2}$$

$$M = EI \frac{d^2y}{dx^2}$$

The above eqn is based only on bending moment.

SIGN CONVENTION



If considered origin at A, x is taken positive towards right and y positive upwards.

→ Deflection is negative (or) it is downwards in both cases.

→ The slope is negative between AC and positive between CB for beam and negative for cantilever throughout.

If the origin is taken at B, x is taken positive towards left and y positive upwards. Then

→ Deflection is negative (or) it is downwards in both cases.

→ The slope is positive between AC and negative between CB for beam and negative for cantilever throughout.

DOUBLE INTEGRATION METHOD

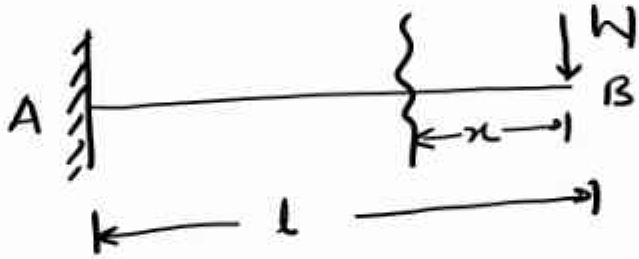
This method is adopted to find out slop & deflection of any beam at any point across cross-section. The equation of elastic curve is integrated twice to obtain the deflection of the beam at any cross-section. The constant of integration are determined by applying the end conditions

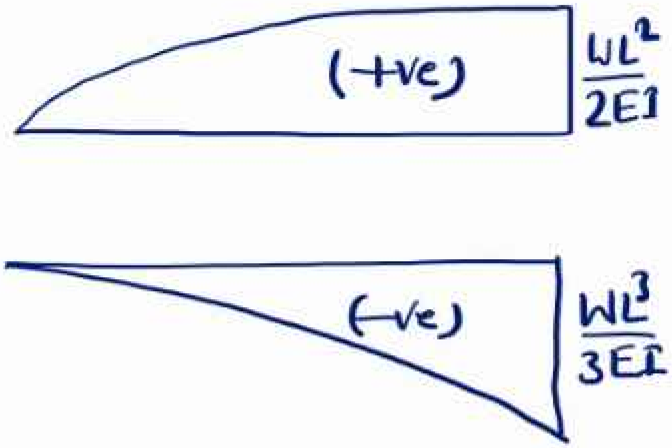
$$EI\frac{dy}{dx} = \int M.dx + C_1$$ {Slope can be calculated at any point}

and $$EI.y = \iint (M.dx) + C_1 x + C_2$$ {Deflection can be determined at any point}

(A) Cantilever beam

(1) Concentrated load at free end





L = length of beam

I = Moment of Inertia

If origin at free end

take a section from B at distance x.

Bending moment at section x

$$= -Wx$$

Here, $M = EI \frac{d^2y}{dx^2}$

$$EI \frac{d^2y}{dx^2} = -Wx \quad \text{——— } (1)$$

Integrate eq^n (1)

$$EI \frac{dy}{dx} = -\frac{Wx^2}{2} + C_1$$

at $x = L$, $\frac{dy}{dx} = 0$.

$$EI \times 0 = -\frac{WL^2}{2} + C_1$$

$$C_1 = \frac{WL^2}{2}$$

$$\therefore EI \frac{dy}{dx} = -\frac{Wx^2}{2} + \frac{WL^2}{2} \quad \text{——— } (2)$$

$$\frac{dy}{dx} = \frac{1}{EI}\left[\frac{WL^2}{2} - \frac{Wx^2}{2}\right]$$

$$\frac{dy}{dx} = \frac{W}{2EI}\left(l^2 - x^2\right)$$

Re-integrate eqn ②

$$EI\,y = -\frac{Wx^3}{6} + \frac{WL^2}{2}x + C_2$$

at $x = L$, $y = 0$

$$\therefore EI \times 0 = -\frac{WL^3}{6} + \frac{WL^3}{2} + C_2$$

$$C_2 = \frac{WL^3}{6} - \frac{WL^3}{2}$$

$$C_2 = \frac{WL^3 - 3WL^3}{6}$$

$$C_2 = -\frac{WL^3}{3}$$

So, $$EI\,y = -\frac{Wx^3}{6} + \frac{WL^2}{2}x - \frac{WL^3}{3} \quad \text{——— ③}$$

at B, $x = 0$, slope & deflection is maximum

from eqn ② $$EI\frac{dy}{dx} = -0 + \frac{WL^2}{2}$$

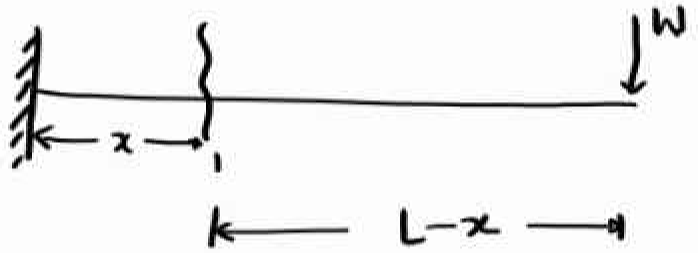
$$\boxed{\frac{dy}{dx} = \frac{WL^2}{2EI}} \rightarrow$$

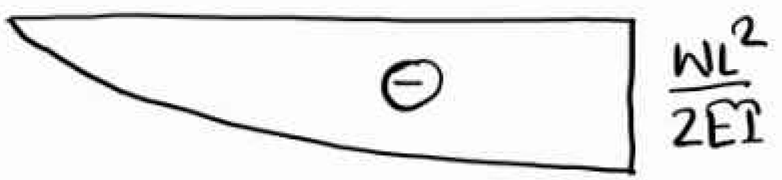
from eq^n (3)

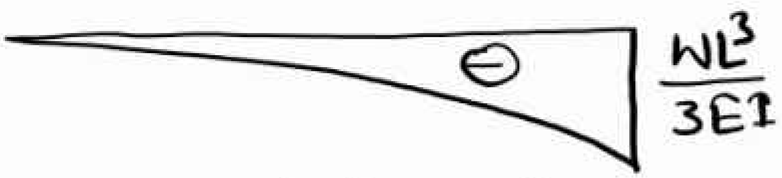
$$EIy = -0+0-\frac{WL^3}{3}$$

$$\boxed{y = \frac{-WL^3}{3EI}}$$

If origin at fixed end







Bending moment at the section $= -W(L-x)$

By using eq^n $\quad M = EI\frac{d^2y}{dx^2}$

$$EI\frac{d^2y}{dx^2} = -W(L-x)$$

Integrating, $EI\frac{dy}{dx} = -W\left(Lx - \frac{x^2}{2}\right) + C_1$

at $x=0$, $\frac{dy}{dx}=0$

$$EI\frac{dy}{dx} = -\left(Lx - \frac{x^2}{2}\right)W \quad \text{——— (7)}$$

Integrating again.

$$EIy = -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) + C_2$$

at $x=0$, $y=0$

$$0 = C_2$$

$$EIy = -W\left(\frac{Lx^2}{2} - \frac{x^3}{6}\right) \quad \text{——— } (2)$$

at $x=L$,

$$EI\frac{dy}{dx} = -W\left(L^2 - \frac{L^2}{2}\right)$$

$$EI\frac{dy}{dx} = -\frac{L^2W}{2}$$

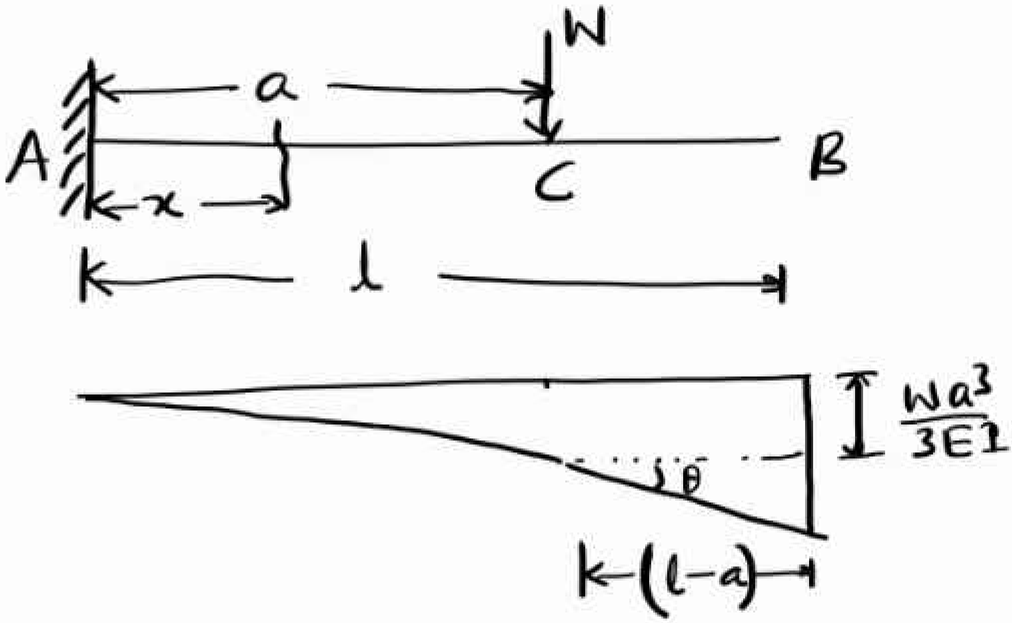
$$\boxed{\frac{dy}{dx} = -\frac{WL^2}{2EI}}$$

$$EIy = -W\left(\frac{L^3}{2} - \frac{L^3}{6}\right)$$

$$= -W \cdot \left(\frac{3L^3 - L^3}{6}\right) = -\frac{WL^3}{3}$$

$$\boxed{y = -\frac{WL^3}{3EI}}$$

(2) Concentrated load not at free end



$$M = -W(a-x)$$

Apply eqn.

$$M = EI\frac{d^2y}{dx^2}$$

$$EI\frac{d^2y}{dx^2} = -W(a-x)$$

(or) $$EI\,y'' = -W(a-x)$$

Integrating once.

$$EI\,y' = -W\left(ax - \frac{x^2}{2}\right) + C_1 \quad \text{——— (1)}$$

$$EI\,y = -W\left(\frac{ax^2}{2} - \frac{x^3}{6}\right) + C_1x + C_2$$

at $x=0$,

$$y'=0 \quad , \quad c_1=0$$

$$EIy' = -W\left(ax-\frac{x^2}{2}\right)$$

at $x=0$

$$c_2=0$$

$$EIy = -W\left(\frac{ax^2}{2}-\frac{x^3}{6}\right)$$

At C., $x=a$

$$y' = -\frac{W}{EI}\left(a\times a-\frac{a^2}{2}\right)$$

$$y' = -\frac{W}{EI}\left(a^2-\frac{a^2}{2}\right)$$

$$\boxed{y'_c = -\frac{Wa^2}{2EI}}$$

$$y = -\frac{W}{EI}\left(\frac{a^3}{2}-\frac{a^3}{6}\right)$$

$$y = -\frac{W}{EI}\left(\frac{3a^3-a^3}{6}\right) = -\frac{W}{EI}\times\frac{2a^3}{6}$$

$$\boxed{y_c = -\frac{Wa^3}{3EI}}$$

Slope at C, $y'_c = -\frac{Wa^2}{2EI}$

$$\tan\theta = \frac{GF}{EG} = -\frac{Wa^2}{2EI} = \frac{GF}{(l-a)}$$

$$GF = -\frac{Wa^2}{2EI}(l-a)$$

Total deflection at B = deflection at C + deflection GF

$$= -\frac{Wa^3}{3EI} - \frac{Wa^2}{2EI}(l-a)$$

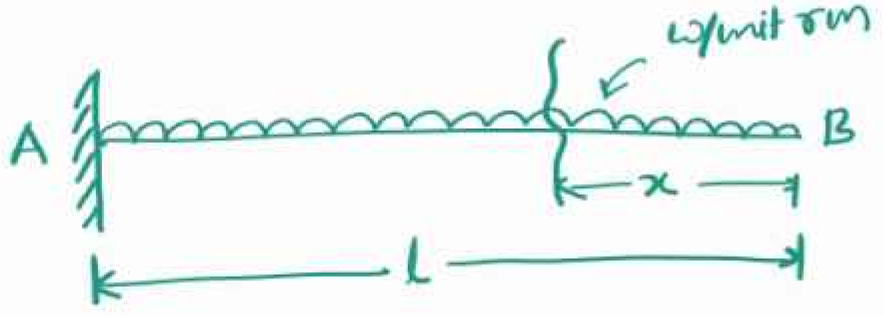
$$= -\frac{Wa^3}{3EI} + \frac{Wa^3}{2EI} - \frac{Wa^2L}{2EI}$$

$$= \frac{-2Wa^3 + 3Wa^3}{6EI} - \frac{Wa^2L}{2EI}$$

$$= \frac{Wa^3}{6EI} - \frac{Wa^2L}{2EI}$$

$$\boxed{y_B = \frac{Wa^2}{2EI}\left(\frac{a}{6} - L\right)}$$

* Uniformly distributed load on whole span



$$M = -\omega x \times \frac{x}{2} = -\frac{\omega x^2}{2}$$

Apply eqⁿ $M = EI \frac{d^2y}{dx^2}$

$$EI \frac{d^2y}{dx^2} = -\frac{\omega x^2}{2} \quad \text{———} \quad (1)$$

Integrate eqⁿ (1)

$$EI \frac{dy}{dx} = -\frac{\omega x^3}{2 \times 3} + C_1$$

$$EI \frac{dy}{dx} = -\frac{\omega x^3}{6} + C_1 \quad \text{———} \quad (2)$$

at $x = L$, $\frac{dy}{dx} = 0$

$$EI \times 0 = -\frac{\omega L^3}{6} + C_1$$

$$C_1 = \frac{\omega L^3}{6}$$

So, $$EI \frac{dy}{dx} = -\frac{\omega x^3}{6} + \frac{\omega L^3}{6} \quad \text{———} \quad (3)$$

Integrate eqⁿ (2) again

$$EI.y = -\frac{\omega x^4}{24} + C_1 x + C_2$$

$$EIy = -\frac{\omega x^4}{24} + \frac{\omega L^3}{6} x + C_2$$

at $x = L$, $y = 0$

$$EI \times 0 = -\frac{\omega L^4}{24} + \frac{\omega L^4}{6} + C_2$$

$$C_2 = \frac{wL^4}{24} - \frac{wL^4}{6} = \frac{wL^4 - 4wL^4}{24}$$

$$C_2 = \frac{-3wL^4}{24} = -\frac{wL^4}{8}$$

$$EIy = -\frac{wx^4}{24} + \frac{wL^3}{6}x - \frac{wL^4}{8} \quad \text{——— (4)}$$

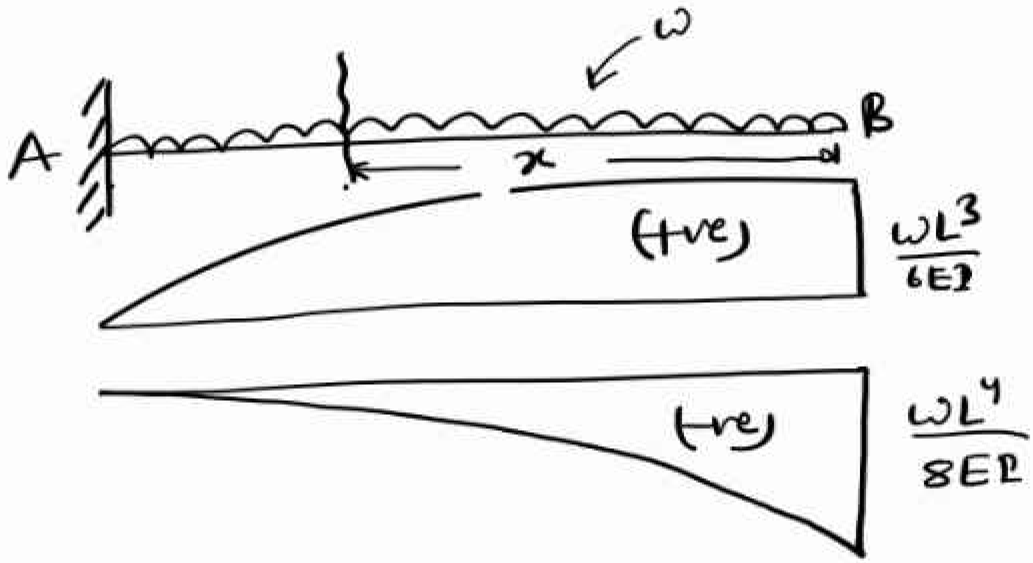
at $x = 0$,

y' is max. & y is maximum.

So,

$$EIy' = \frac{wL^3}{6} \Rightarrow y' = \frac{wL^3}{6EI}$$

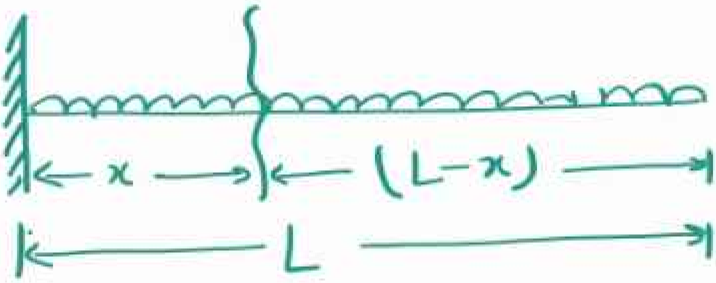
$$EIy = \frac{-wL^4}{8} \Rightarrow y = -\frac{wL^4}{8EI}$$

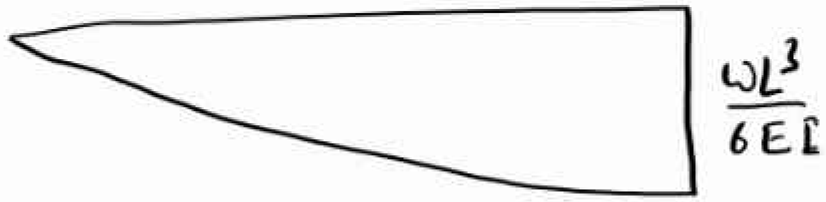


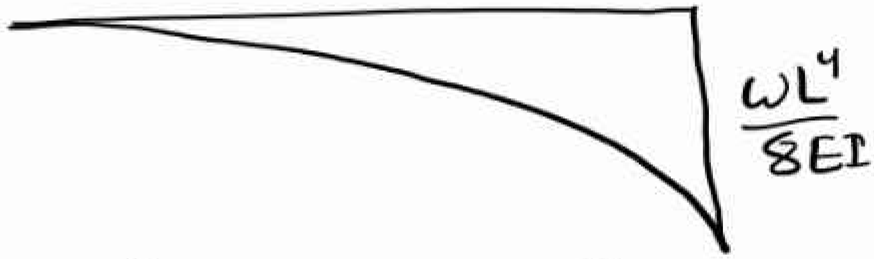
If origin is taken at A.

$$M = -w(L-x) \times \frac{(L-x)}{2}$$

$$M = -w\frac{(L-x)^2}{2}$$







$$EI\frac{d^2y}{dx^2} = -\frac{\omega(L-x)^2}{2} \quad \text{——— (1)}$$

$$= -\frac{\omega}{2}\left(L^2+x^2-2Lx\right)$$

<u>Integrate</u> eqn (1)

$$EI\frac{dy}{dx} = -\frac{\omega}{2}\left(L^2x+\frac{x^3}{3}-Lx^2\right)+c_1$$

at $x=0$

$$EI\times 0 = c_1 \Rightarrow c_1 = 0$$

$$EI\frac{dy}{dx} = -\frac{\omega}{2}\left(L^2x+\frac{x^3}{3}-Lx^2\right) \quad \text{——— (2)}$$

Integrate again slope eqn.

$$EI\cdot y = -\frac{\omega}{2}\left(\frac{L^2x^2}{2}+\frac{x^4}{12}-\frac{Lx^3}{3}\right)+c_1x+c_2$$

at $x=0$,

$$EI \times 0 = -\frac{\omega}{2}(0) + C_1 \times 0 + C_2$$

$$C_2 = 0$$

$$EIy = -\frac{\omega}{2}\left(\frac{L^2x^2}{2} + \frac{x^4}{12} - \frac{Lx^3}{3}\right) \text{ ——— } (3)$$

at $x=L$, $\frac{dy}{dx}$ is maximum.

$$EI\frac{dy}{dx} = -\frac{\omega}{2}\left(L^3 + \frac{L^3}{3} - L^3\right)$$

$$= -\frac{\omega}{2} \times \frac{L^3}{3}$$

$$\frac{dy}{dx} = -\frac{\omega L^3}{6EI}$$

$$\boxed{y' = -\frac{\omega L^3}{6EI}}$$

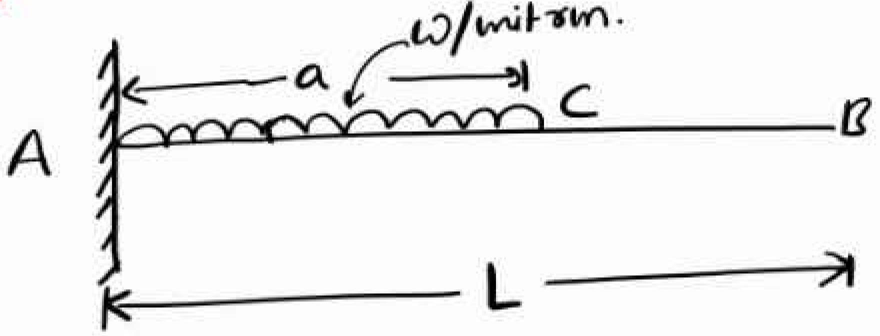
at $x=L$, y is maximum.

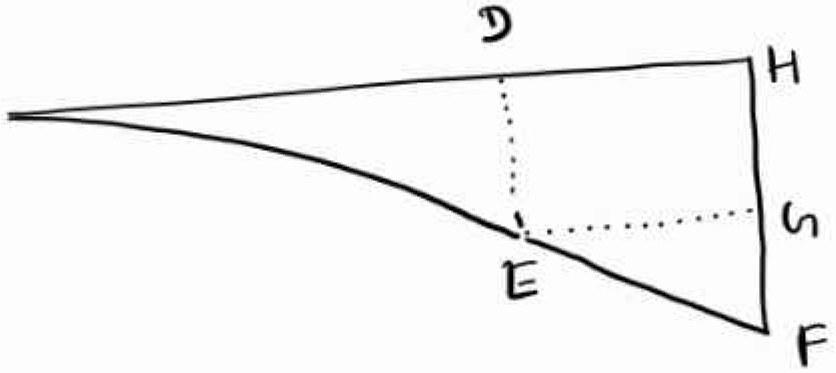
$$EIy = -\frac{\omega}{2}\left(\frac{L^4}{2} + \frac{L^4}{12} - \frac{L^4}{3}\right)$$

$$= -\frac{\omega}{2}\left(\frac{7L^4}{12} - \frac{L^4}{3}\right) = -\frac{\omega}{2}\,\frac{3L^4}{124}$$

$$\boxed{y = -\frac{\omega L^4}{8EI}}$$

Uniformly distributed load on a part of span from fixed end.





Slope at C and deflection at C.

$$y' = -\frac{wa^3}{6EI}$$

$$y_C = \frac{-wa^4}{8EI}$$

$$y'_C = \frac{GF}{EG}$$

$$-\frac{wa^3}{6EI} = \frac{GF}{(L-a)}$$

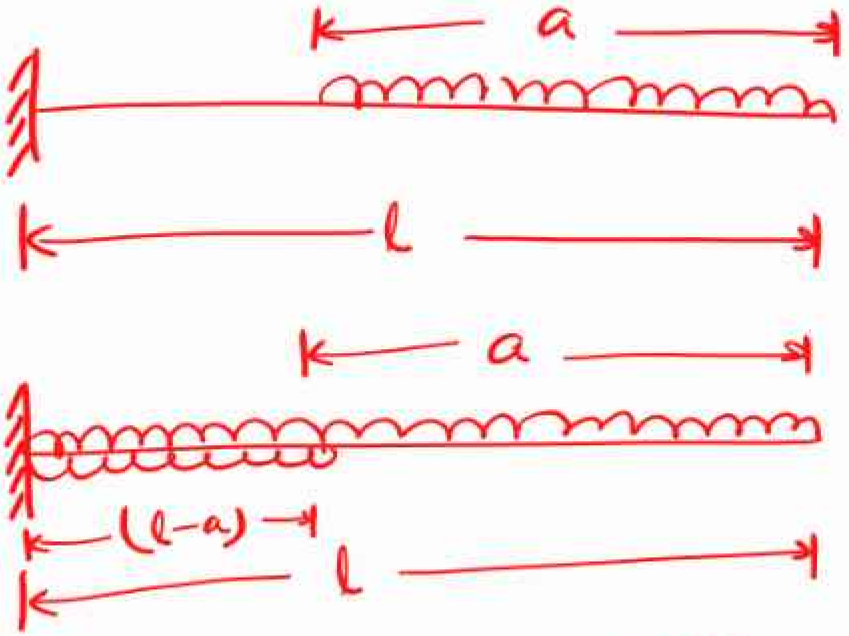
$$GF = -\frac{wa^3}{6EI}(L-a)$$

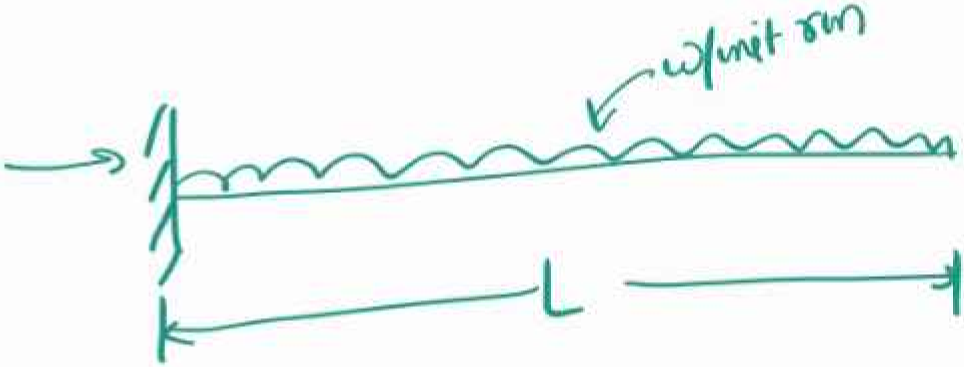
deflection at B

$$y_B = HG + GF = DE + GF$$

$$= -\frac{wa^4}{8EI} + \left\{ -\frac{wa^3}{6EI}(l-a) \right\}$$

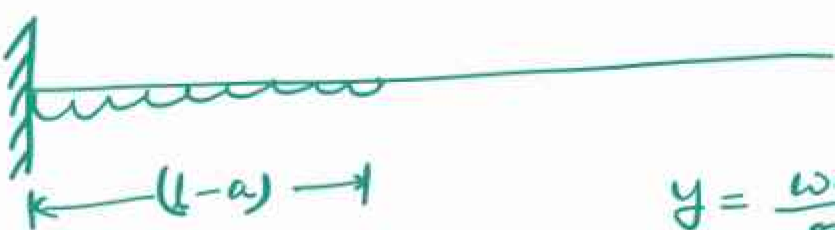
$$\boxed{y_B = -\frac{wa^4}{8EI} - \frac{wa^3}{6EI}(l-a)}$$





$$\frac{dy}{dx} = \frac{wl^3}{6EI}$$

$$y = \frac{wl^4}{8EI}$$



$$\frac{dy}{dx} = -\frac{w(l-a)^3}{6EI}$$

$$y = \frac{wa^4}{8EI} - \frac{wa^3}{6EI}(l-a)$$

$$\text{Net slope} = \frac{wl^3}{6EI} + \left\{ -\frac{w(l-a)^3}{6EI} \right\}$$

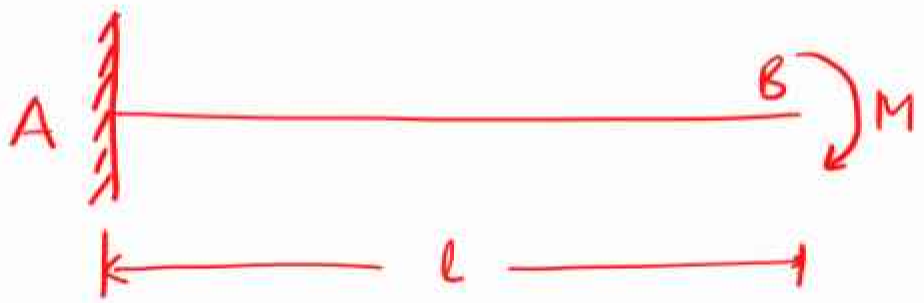
$$\boxed{\text{Net slope} = \frac{wl^3}{6EI} - \frac{w}{6EI}(l-a)^3}$$

$$\text{Net deflection} = \frac{wl^4}{8EI} - \left[\frac{wa^4}{8EI} + \frac{wa^3}{6EI}(l-a) \right]$$

$$= \frac{wl^4}{8EI} - \frac{wa^3}{2EI}\left(\frac{a}{4} + \frac{l-a}{3} \right)$$

$$\boxed{\text{Net deflection} = \frac{wl^4}{8EI} - \frac{wa^3}{2EI}\left(\frac{a}{4} + \frac{l}{3} - \frac{a}{3} \right)}$$

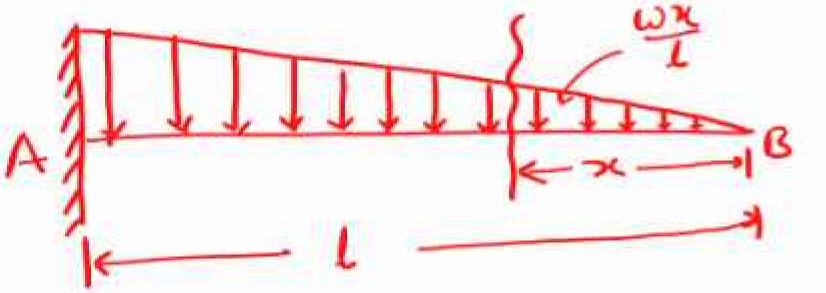
Find the slope & deflection if a couple is applied at the free end of a cantilever beam



$$\frac{dy}{dx} = -\frac{M}{EI}x$$

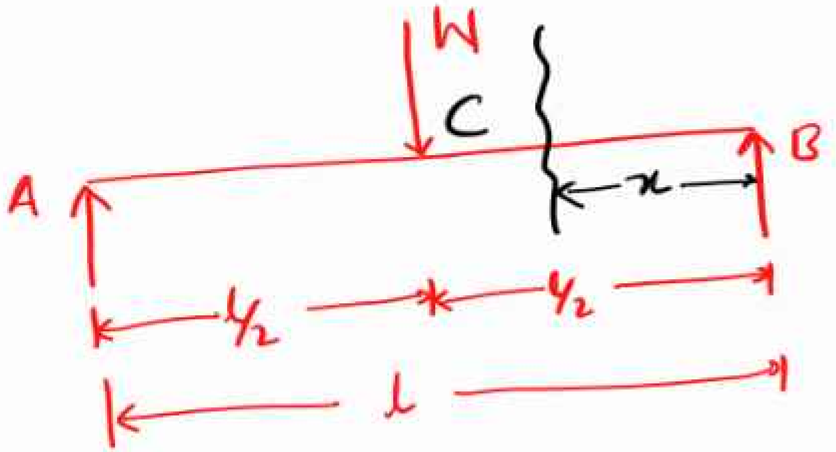
$$y = -\frac{M}{2EI}x$$

Distributed load of varying load intensity, zero at the free end.



$$M_c = \frac{wx^3}{6L}$$

#



$$R_A = R_B = \frac{W}{2}$$

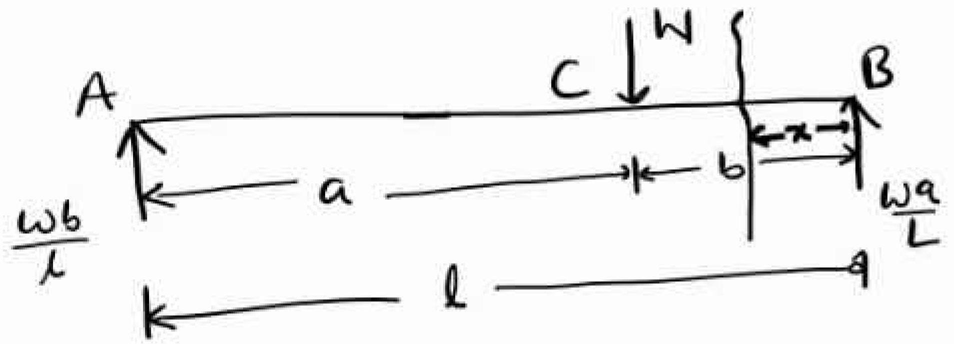
$$M = \frac{W}{2}x$$

at $x=0$, $\frac{dy}{dx} = \frac{-WL^2}{16EI}$

at $x = L/2$

$$y_c = -\frac{WL^3}{48EI}$$

Find the slope & deflection by using a ecentric concentrated load on simply supported beam.



$$M_{BC} = \frac{Wa}{L} x \qquad\qquad R_A = \frac{Wb}{l}$$

$$M_{CA} = \frac{Wa}{L} x - W(x-b) \qquad\qquad R_B = \frac{Wa}{l}$$

For section BC

$$EI \frac{d^2y}{dx^2} = \frac{wa}{L} x \quad \text{———} ①$$

Integrate eqn ①

$$EI \frac{dy}{dx} = \frac{Wa}{L} \frac{x^2}{2} + C_1 \quad \text{———} ②$$

Integrate eqn ② again

$$EIy = \frac{Wa}{L} \frac{x^3}{6} + C_1 x + C_2 \quad \text{———} ③$$

at $x = b$, $\frac{dy}{dx} = 0$

$$EI \times 0 = \frac{Wab^2}{2L} + C_1$$

$$C_1 = -\frac{Wab^2}{2L}$$

at $x=0,\ y=0$

$$EI \times 0 = \frac{Wa}{L} \times 0 + C_1 \times 0 + C_2$$

$$C_2 = 0$$

For segment CA

$$EI\frac{d^2y}{dx^2} = \frac{Wa}{L}x - W(x-b) \quad \text{---} \ (4)$$

Integrate once

$$EI\frac{dy}{dx} = \frac{Wa}{L}\frac{x^2}{2} - \frac{W(x-b)^2}{2} + C_3 \quad \text{---} \ (5)$$

Integrate eqn (5)

$$EI \cdot y = \frac{Wax^3}{6L} - \frac{W(x-b)^3}{6} + C_3x + C_4 \quad \text{---} \ (6)$$

from (2) & (5)

Slope at C will be equal

$$\cancel{\frac{Wax^2}{2L}} + C_1 = \cancel{\frac{Wa}{L}\frac{x^2}{2}} - \frac{W(x-b)^2}{2} + C_3$$

$$C_1 = -\frac{W(x-b)^2}{2} + C_3$$

at, $x=b$

$$C_1 = C_3$$

from eqn (3) & (5)

deflection at C from (3) = deflection at C from (6)

$$\cancel{\frac{Wa}{L}\frac{x^3}{6}} + C_1x + C_2 = \cancel{\frac{Wax^3}{6L}} - \frac{W(x-b)^3}{6} + C_3x + C_4$$

at $x = b$,

$$c_1 b + 0 = -0 + c_3 b + c_4$$

$$c_4 = 0$$

from eqⁿ (6)

at $x = L$, $y = 0$

$$EI \times 0 = \frac{WaL^3}{6L} - \frac{W(L-b)^3}{6} + c_3 L + c_4 = 0$$

$$\Rightarrow \quad \frac{WaL^2}{6} - \frac{W(L-b)^3}{6} + c_3 L = 0$$

$$c_3 = \frac{W(L-b)^3}{6L} - \frac{WaL^2}{6L}$$

$$= \frac{Wa^3}{6L} - \frac{WaL^2}{6L} = \frac{Wa}{6L}(a^2 - L^2)$$

$$c_3 = -\frac{Wa}{6L}(L^2 - a^2)$$

$$c_1 = -\frac{Wa}{6L}(L^2 - a^2)$$

$$EIy = \frac{Wa}{L}\frac{x^3}{6} + c_1 x + c_2$$

$$EIy = \frac{Wa}{L}\frac{x^3}{6} - \frac{Wa}{6L}(L^2 - a^2)x \quad \text{—— (7)} \; [\text{for } x < b]$$

from eqⁿ (6)

$$EIy = \frac{Wax^3}{6L} - \frac{W(x-b)^3}{6} + c_3 x + c_4$$

$$EIy = \frac{Wax^3}{6L} - \frac{W(x-b)^3}{6} - \frac{Wa}{6L}(L^2 - a^2)x \quad \text{—— (8)} \; [\text{for } x > b]$$

The value of x for maximum deflection can be obtained.

let $a > b$.

from eqn ①

$$EIy' = \frac{wax^2}{2L} - \frac{Wa}{6l}(l^2 - a^2)$$

at maximum deflection, slope = 0

(or) $$0 = \frac{Wax^2}{2L} - \frac{Wa}{6L}(l^2 - a^2)$$

$$x^2 - \frac{1}{3}(l^2 - a^2) = 0$$

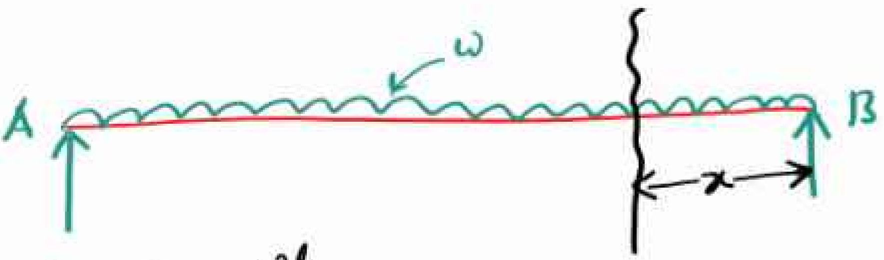
$$x^2 = \frac{(l^2 - a^2)}{3}$$

$$\boxed{x = \sqrt{\frac{l^2 - a^2}{3}}}$$

Thus maximum deflection

$$\boxed{EIy = \frac{Wa}{6L}\left(\sqrt{\frac{l^2 - a^2}{3}}\right)^3 - \frac{Wa}{6L}(L^2 - a^3)\left(\sqrt{\frac{l^2 - a^2}{3}}\right)}$$

#



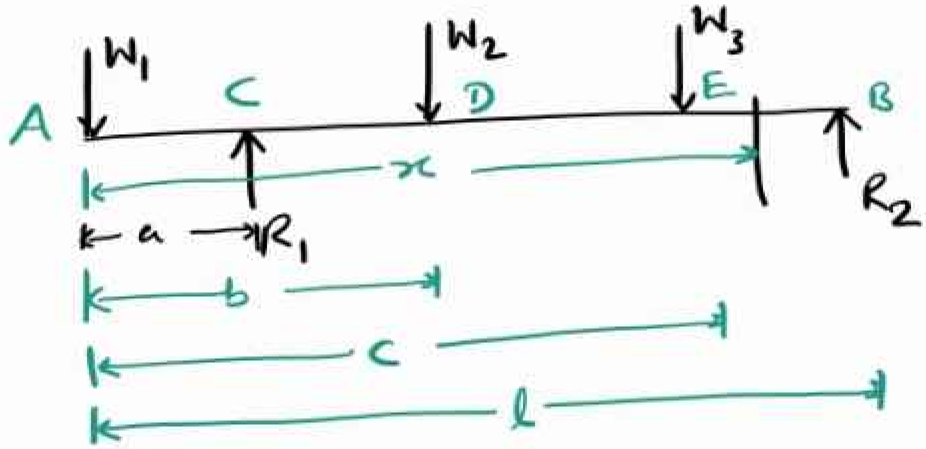
$$R_a = R_b = \frac{wl}{2}$$

$$EI\frac{d^2y}{dx^2} = \frac{wlx}{2} - \frac{wx^2}{2}$$

$$\left(\frac{dy}{dx}\right)_{max} = \frac{wl^3}{24EI}$$

MACALAY'S METHOD

While applying the double integration method, a separate expression for the bending moment is needed to be written for each section of the beam.



Consider a simply supported beam AB of length l as shown in figure. Taking A as origin and writting the expression for the bending moment.

$$EI \frac{d^2y}{dx^2} = M = -W_1 x \,|\, + R_1(x-a) \,|\, - W_2(x-b) - W_3(x-c)$$

In the above expression, there are saperation lines —

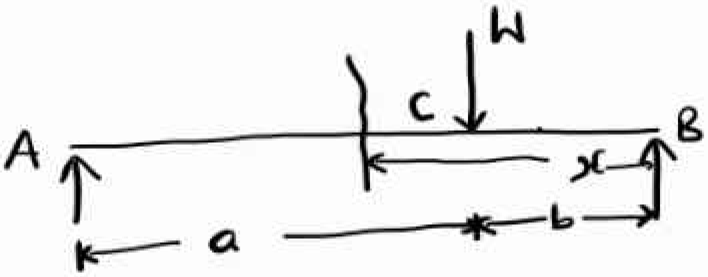
* The portion to the left of the first separation line is valid for the portion AC.
* The portion to the left of the second separation line is valid for the portion CD.
* The portion to the left to the third separation line is valid for the portion DE.
* The whole of the expression is valid for the portion EB.

so, $$EI\frac{dy}{dx} = -W_1\frac{x^2}{2} + C_1 \Big| + \frac{R_1}{2}(x-b)^2 \Big| - \frac{W_2}{2}(x-b)^2 \Big| - \frac{W_3}{2}(x-c)^2$$

and $$EIy = -W_1\frac{x^3}{6} + C_1 x + C_2 \Big| + \frac{R_1}{6}(x-a)^3 \Big| - \frac{W_2}{6}(x-b)^3 \Big| - \frac{W_3}{6}(x-c)^3$$

Constant C_1 & C_2 are evaluated from end conditions.

Simply supported beam having an eccentric load W.



$$R_A = \frac{Wb}{l}, \quad R_B = \frac{Wa}{l}$$

$$EI\frac{d^2y}{dx^2} = \underbrace{\frac{Wa}{l}x}_{BC} \Big| \underbrace{- W(x-b)}_{CA} \qquad \text{——— (1)}$$

Integrating.

$$EI\frac{dy}{dx} = \frac{Wax^2}{2l} + C_1 \Big| - \frac{W(x-b)^2}{2} \qquad \text{——— (2)}$$

Integrating again

$$EIy = \frac{Wax^3}{6l} + C_1 x + C_2 \Big| - \frac{W(x-b)^3}{6} \qquad \text{——— (3)}$$

at B, $x = 0$, $y = 0$

The eqn of the elastic curve for portion BC in eqn (3)

$$EI \times 0 = 0 + 0 + c_2$$

$$c_2 = 0$$

at A, $x = l$, $y = 0$

The whole eqn is valid for the portion CA. from eqn ③

$$EI \times 0 = \frac{Wal^3}{6l} + c_1 l + 0 - \frac{W(l-b)^3}{6}$$

$$= \frac{Wal^2}{6} - \frac{Wa^3}{6} + c_1 l$$

$$c_1 l = -\frac{Wal^2}{6} + \frac{Wa^3}{6}$$

$$= -\frac{Wa}{6}(l^2 - a^2)$$

$$c_1 = -\frac{Wa}{6l}(l^2 - a^2)$$

Thus, the slope & deflection is given by.

$$EI\frac{dy}{dx} = \frac{Wax^2}{2l} - \frac{Wa}{6l}(l^2 - a^2) \Big| - \frac{W(x-b)^2}{2}$$

$$\text{and } EIy = \frac{Wax^3}{6l} - \frac{Wax}{6l}(l^2 - a^2) \Big| - \frac{W(x-b)^3}{6}$$

at $x = b$,

$$EIy = \frac{Wab^3}{6l} - \frac{Wab}{6l}(l^2 - a^2)$$

$$= -\frac{Wab}{6l}(-b^2 + l^2 - a^2)$$

$$= -\frac{Wab}{6l}\left\{(a+b)^2 - a^2 - b^2\right\}$$

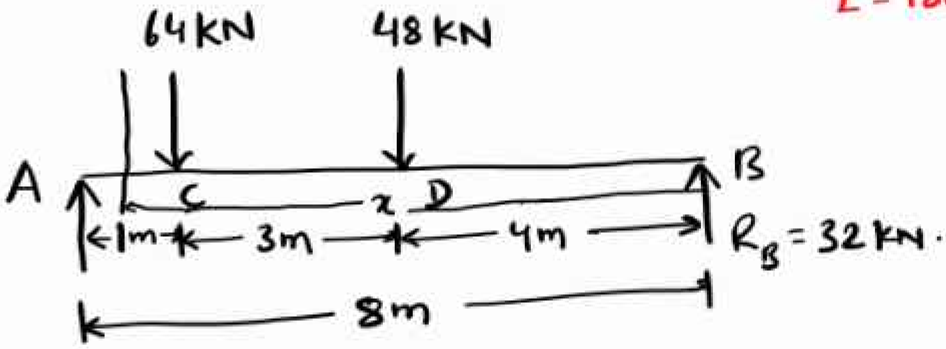
$$= -\frac{Wab}{6l}\left\{\cancel{a^2}+\cancel{b^2}+2ab-\cancel{a^2}-\cancel{b^2}\right\}$$

$$EIy = -\frac{Wa^2b^2}{6l}$$

$$\boxed{y = -\frac{Wa^2b^2}{6EIl}}$$

Problem A simply supported beam of 8m length carries two point loads of 64 kN and 48 kN at 1m and 4m respectively from left hand end. Find deflection under each load and the maximum deflection. $E = 210\ GPa$

$I = 180 \times 10^6\ mm^4$



$$R_A = 80\,kN, \quad R_B = 32\,kN.$$

$$M = 32x \,\big|\, -48(x-4) \,\big|\, -64(x-7)$$

$$EI\frac{d^2y}{dx^2} = 32x \,\big|\, -48(x-4) \,\big|\, -64(x-7) \quad \text{———} \quad (1)$$

Integrating eqⁿ ①

$$EI\frac{dy}{dx} = \frac{32x^2}{2} + C_1 \left| -\frac{48(x-4)^2}{2} \right| - \frac{64(x-7)^2}{2} \quad \text{---} (2)$$

Integrating again eqⁿ ②

$$EIy = \frac{32x^3}{6} + C_1x + C_2 \left| - \frac{48(x-4)^3}{6} \right| - \frac{64(x-7)^3}{6}$$

at B, $x=0$, $y=0$. The eqⁿ of elastic curve <u>BD.</u>

$$EIy = 0 + 0 + C_2$$

$$C_2 = 0$$

at $x = 8\,m$, $y = 0$, The whole eqⁿ is valid for <u>CA</u>

$$0 = \frac{32\times8^3}{6} + C_1\times8 + 0 - \frac{48(8-4)^3}{6} - \frac{64(8-7)^3}{6}$$

$$\frac{8192}{3} + 8C_1 - 8\times64 - \frac{64}{6} = 0$$

$$8C_1 = 8\times64 + \frac{64}{6} - \frac{8192}{3}$$

$$C_1 = -276$$

<u>deflection under load 48 KN</u>

$$EIy = \frac{32\times4\times4\times4}{6} - 276\times4 = -\frac{2288}{3}$$

$$EIy = -\frac{2288}{3}$$

$$y = -\frac{2288 \times 10^{12}}{3 \times 210 \times 10^3 \times 180 \times 10^6}$$

$$y = -20.176 \text{ mm}$$

deflection under load 64 KN

$$EIy = \frac{32 \times 7^3}{6} - 276 \times 7 - 48 \times \frac{3^3}{6}$$

$$= \frac{32 \times 7^3}{6} - 276 \times 7 - \frac{48 \times 27}{6}$$

$$EIy = 1829.33 - 1932 - 216$$

$$y = -\frac{318.67 \times 10^{12}}{210 \times 10^3 \times 180 \times 10^6}$$

$$y = -8.43 \text{ mm}$$

The maximum deflection will be DC, so, at maximum deflection, slope will be zero.

from eqn 2

$$EI \times 0 = 16x^2 - 276 - 24(x-4)^2$$

$$16x^2 - 276 - 24(x^2+16-8x) = 0$$

$$16x^2 - 276 - 24x^2 - 24\times16 + 8\times24x = 0$$

$$-8x^2 + 8\times24x - 660 = 0$$

$$x^2 - 24x + 82.5 = 0$$

$$x = \frac{24 \pm \sqrt{(24)^2 - 4\times1\times82.5}}{2\times1}$$

$$x = 19.84\,m, \; 4.157\,m.$$

Now, max deflection will be at 4.157m distance from End B.

$$EI\,y_{max} = \frac{32x^3}{6} - 276x - 8(x-4)^3$$

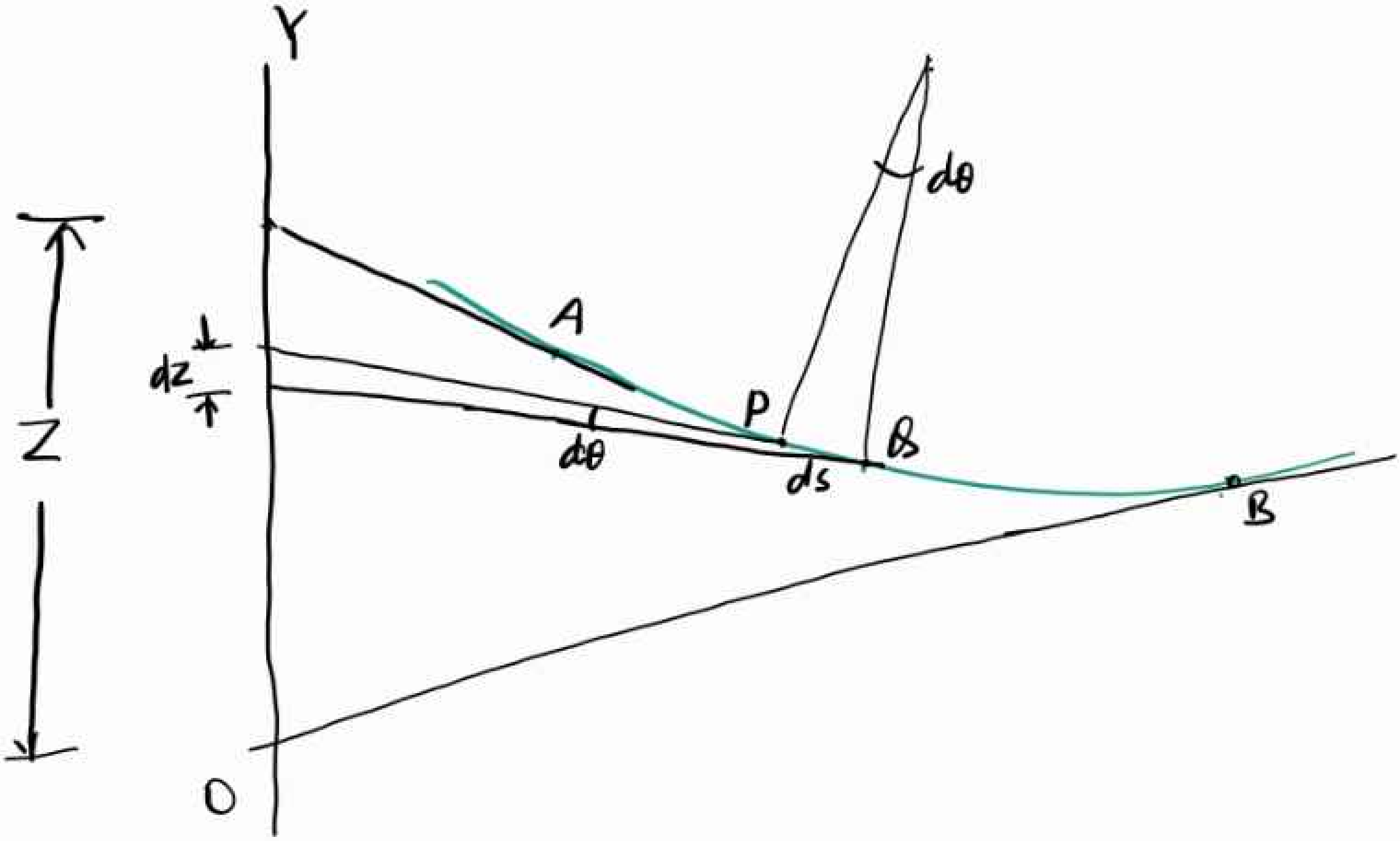
put $x = 4.157m$.

$$EI\,y_{max} = \frac{32\times4.157^3}{6} - 276\times4.157 - 8\times0.157^3$$

$$= \frac{-764.24\times10^{12}}{210\times10^3\times180\times10^6}$$

$$\boxed{y_{max} = 20.22\,mm}$$

MOMENT-AREA METHOD (MOHR'S THEOREMS)



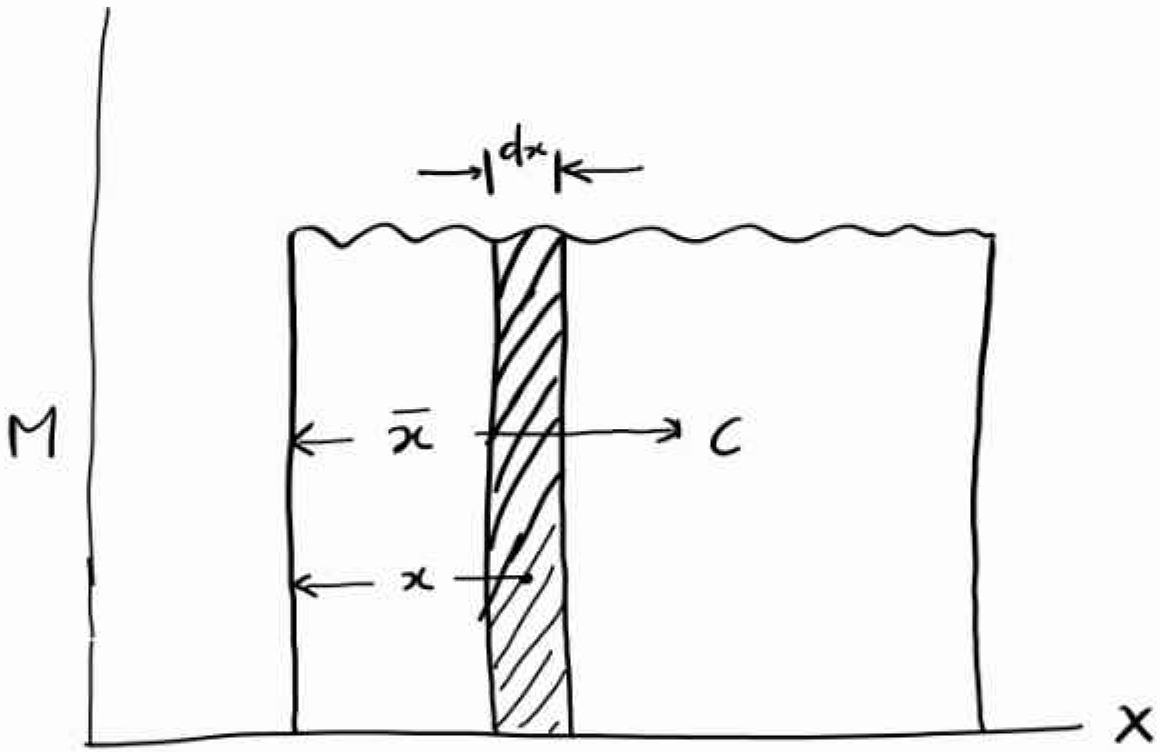


Figure shows the shape of a deflected beam between two choosen points A and B. In order to find the difference in slopes of theses points, consider two sections P & Q at a very small distance ds along the curve.

After bending, the normal section made an angle $d\theta$ at the centre of curvature.

$$ds = R\,d\theta$$

$$d\theta = \frac{ds}{R} = ds \cdot \frac{M}{EI}$$

Since the curvature is very small, $ds \simeq dx$

(or) $d\theta = \frac{M}{EI}\,dx$

As $d\theta$ is equal to the change of slope between points P & Q, the change of slope between A & B is given by

$$\theta_A^B = \sum_A^B d\theta = \int_A^B \frac{M}{EI} \cdot dx$$

In the above expression, $M \cdot dx$ is the area of bending moment diagram between P & Q

Thus, $\theta_A^B = \frac{1}{EI} \times$ Area of bending moment diagram between A & B.

From the above, Mohr's first moment-area theorem can be stated as.

The difference of slope between any two points on an elastic curve of a beam is equal to the net area of the bending moment diagram between these two points divided by EI.

$$\left[\frac{dy}{dx}\right]_A^B = \int_A^B \frac{M\,dx}{EI} \quad \text{(or)} \quad \left[\frac{dy}{dx}\right]_B - \left[\frac{dy}{dx}\right]_A = \frac{A}{EI}$$

Now, consider a vertical line OY given in figure. Let the tangents at P & Q to the elastic line cut off an intercept dz on this line. The angle between the tangents will be also equal to $d\theta$.

So, $d\theta = \frac{M}{EI} dx$

Multiplying both sides with x

$$d\theta \cdot x = \frac{M}{EI} x \, dx$$

(or) $$dz = \frac{(M\,dx)}{EI} x$$

The deflection due to bending of all elemental portions between A & B will be given by integrating the above eqn.

$$\boxed{Z = \int \frac{(M \cdot dx)\, x}{EI} = \frac{A\bar{x}}{EI}}$$

Z = Intercept on the vertical line.

A = Area of the bending moment diagram between A & B.

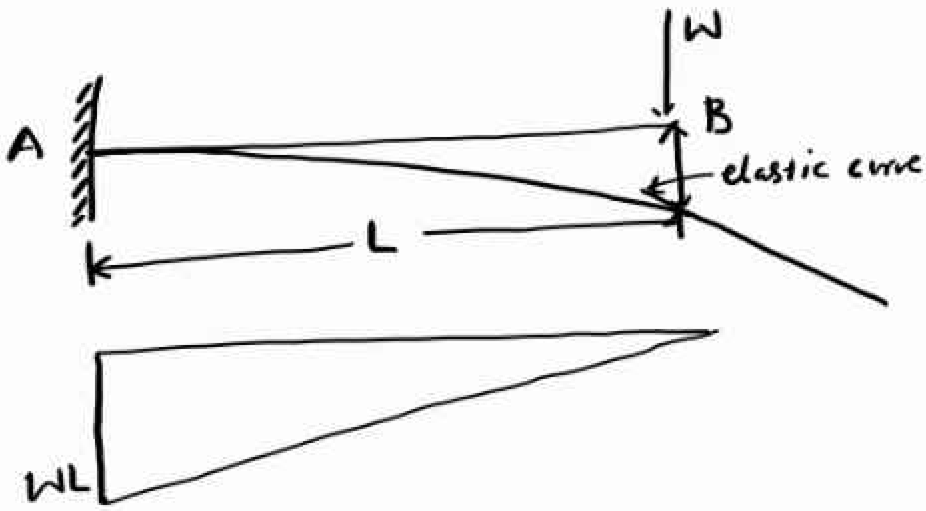
$\bar{x}$ = distance of the centroid of the area from the vertical line OY.

The above eqn leads to the statement of Mohr's second theorem.

The intercept on a given line between the tangents to the elastic curve of a beam at any two points is equal to the net moment taken about that line of the area of the bending moment diagrams between the two points divided by EI.

Examples:

Cantilever beam with a concentrated load at the free end.



Area of bending moment diagram, $A = -\frac{1}{2} L \cdot WL = -\frac{WL^2}{2}$

According to Mohr's first moment area theorem

Differences of slopes between A & B $= \dfrac{\text{Area of B.M. diagram b/w A \& B}}{EI}$

As, slope at A is zero.

$$\therefore \text{Slope at } B = \frac{A}{EI} = -\frac{WL^2}{2EI}$$

Deflection B = Intercept on a vertical line at B between tangents at A & B on elastic curve.

According to Mohr's second moment area theorem

Intercept on a vertical line at B made by tangents at A & B on elastic curve.

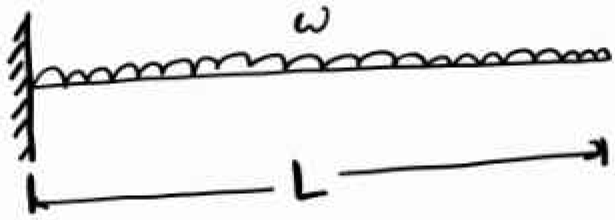
$$= \frac{(\text{Net moment of area of BM diagram between A \& B about B})}{EI}$$

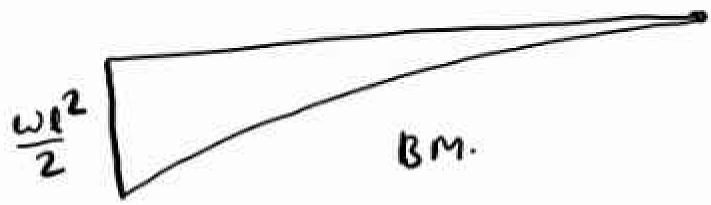
$\therefore$ Deflection at B $= \dfrac{A\bar{x}}{EI}$

Where, $\bar{x}$ = distance of the centroid of area from B.

$$\boxed{\text{deflection} = -\frac{Wl^2}{2EI}\cdot\frac{2L}{3} = -\frac{WL^3}{3EI}}$$

Cantilever beam with uniformly distributed load





$$A = -\frac{1}{3}L\cdot\frac{\omega L^2}{2} = -\frac{WL^3}{6}$$

slope at free end

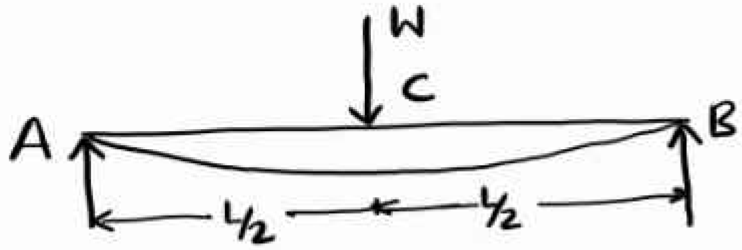
$$= \frac{A}{EI} = -\frac{\omega L^3}{6EI}$$

$$\text{deflection} = \frac{A}{EI}\cdot\bar{x}$$

$$= -\frac{\omega L^3}{6EI}\cdot\frac{3L}{4} = -\frac{\omega L^4}{8EI}$$

$$\boxed{\text{deflection} = \frac{-\omega L^4}{8EI}}$$

Simply supported beam with concentrated load at the mid span



Bending moment diagram

Area of BM diagram between A & C

$$= \frac{1}{2} \times \frac{L}{2} \times \frac{WL}{4}$$

$$= \frac{WL^2}{16}$$

According to Mohr's first moment area theorem

Difference of slopes between A & C = $\frac{\text{Area of BM diagram b/w A \& C}}{EI}$

As, slope at C = 0,

Slope at A (or) B $= \frac{A}{EI} = \frac{WL^2}{16EI}$

Deflection at C w.r.t. A = deflection at A w.r.t. C

= Intercept on a vertical line at B between tangents at B & C on elastic curve

According to Mohr's second moment area theorem.

Intercept on a vertical line at B made by tangents at B & C on elastic curve.

$= \left(\text{Net moment of area of B.M. diagram b/w B \& C about B}\right) / EI$

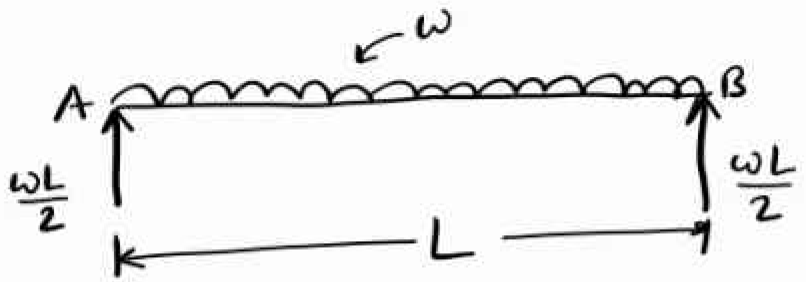
$\therefore$ deflection at C: $\dfrac{A\bar{x}}{EI}$

$$\bar{x} = \frac{2}{3}\frac{L}{2} = \frac{L}{3}$$

$$\text{deflection at } C = \frac{WL^2}{16} \times \frac{L}{3EI}$$

$$\boxed{\text{deflection at } C = \frac{WL^3}{48EI}}$$

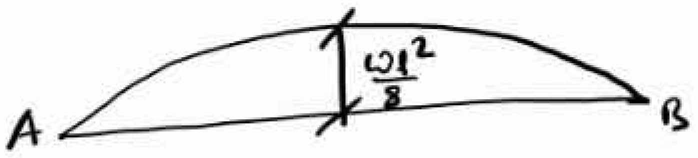
Simply supported beam with uniformly distributed load



Area of BM diagram

$$= \frac{2}{3} \cdot \frac{L}{2} \times \frac{wL^2}{8}$$

$$= \frac{wL^3}{24}$$



$$\text{Slope at } B = -\frac{A}{EI} = -\frac{wL^3}{24EI}$$

Maximum deflection will be at the mid point of line AB.

$$\text{deflection at mid point} = \frac{A\bar{x}}{EI}$$

$$\bar{x} = \frac{5}{8} \cdot \frac{L}{2} = \frac{5L}{16}$$

$$\text{Deflection } (y_{mid}) = -\frac{wL^3}{24EI} \times \frac{5L}{16}$$

$$\boxed{y_{mid} = -\frac{5wL^4}{384EI}}$$

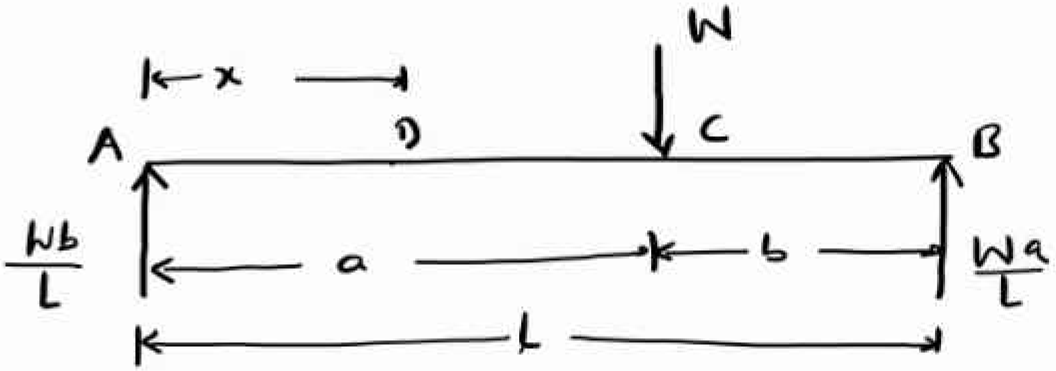
A simply supported beam of length L carries a Concentrated load W at a distance 'a' from end A. Determine.

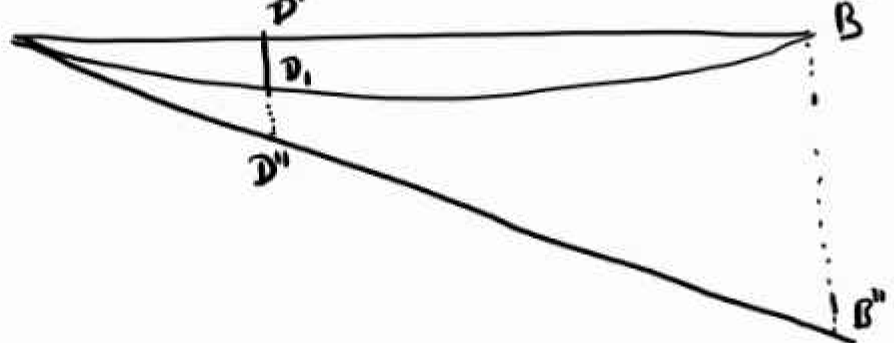
(a) Deflection at the mid span

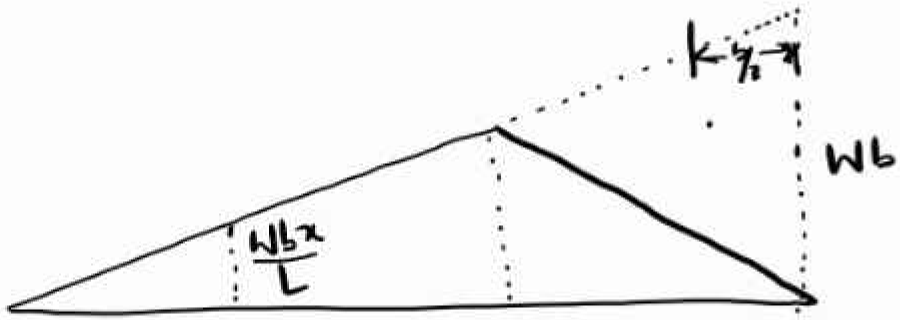
(b) deflection under the load

(c) Maximum deflection

Soln.







$$R_A = \frac{Wb}{l}, \quad R_B = \frac{Wa}{L}$$

Apply Mohr's 2nd moment area theorem

Consider a section D at a distance x from A between A & C.

Deflection at D,

$$D'D_1 = D'D'' - D_1D''$$

$$\frac{D'D''}{x} = \frac{B'B''}{L}$$

$$D'D'' = BB''. \frac{x}{L}$$

But $B'B''$ is the intercept made by tangents at A & B on the elastic curve on a vertical line at B.

According to Mohr's second moment-area theorem, it must be equal to the net moment of area of bending moment diagram between A & B about B divided by EI

$$B'B'' = \frac{1}{EI}\left(\frac{1}{2} Wb \times L \times \frac{L}{3} - \frac{1}{2} Wb \times b \times \frac{b}{3} \right)$$

$$= \frac{Wb}{6EI}\left(l^2 - b^2\right)$$

$$\therefore D'D'' = \frac{Wb}{6EI}(l^2-b^2)\frac{x}{L}$$

$$D'D'' = \frac{Wbx}{6EIL}(L^2-b^2)$$

Now, $D''D_1$ is the intercept made by tangents at D and A on elastic curve on a vertical line at D.

According to Mohr's second moment area theorem, it must be equal to the net moment of area of bending moment diagram between A & D about D divided by EI.

$$D''D_1 = \frac{1}{EI}\left(\frac{1}{2}\frac{Wbx}{l}\cdot x\cdot\frac{x}{3}\right) = \frac{Wbx^3}{6EIL}$$

$$\therefore \text{Deflection at } D = \frac{Wbx}{6EIL}(L^2-b^2) - \frac{Wbx^3}{6EIL}$$

$$= \frac{Wbx}{6EIL}(L^2-b^2-x^2) \quad \text{—— (1)}$$

<u>Deflection at mid span & under the load</u>

<u>Deflection at mid span</u> $(x = L/2)$

$$= \frac{Wb}{6EIL}\left(\frac{L}{2}\right)\left(L^2-b^2-\frac{L^2}{4}\right) = \frac{Wb}{48EI}(3L^2-4b^2)$$

Deflection under the load ($x = a$)

$$= \frac{Wba}{6EIL}(L^2 - b^2 - a^2) = \frac{Wab}{6EIL}\left[(a+b)^2 - b^2 - a^2\right]$$

$$= \frac{Wa^2b^2}{3EIL}$$

Maximum deflection

For max. deflection, differentiate 1 w.r.t. x and equate to zero.

$$\frac{d}{dx}(l^2x - b^2x - x^3) = 0$$

$$l^2 - b^2 - 3x^2 = 0$$

$$x = \sqrt{\frac{l^2 - b^2}{3}}$$

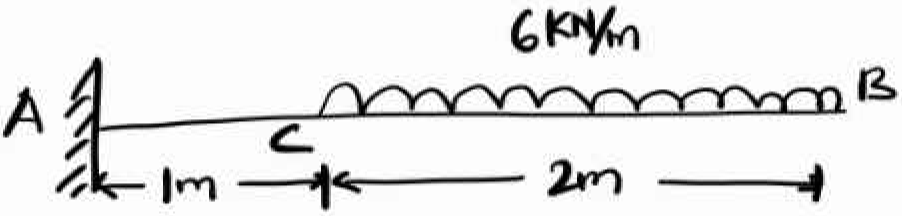
$$\text{Max. deflection} = \frac{Wb}{6EIL}\sqrt{\frac{L^2 - b^2}{3}}\left[L^2 - b^2 - \frac{l^2 - b^2}{3}\right]$$

$$= \frac{Wb(L^2 - b^2)^{3/2}}{9\sqrt{3}\,EIL}$$

An 80mm wide and 180mm deep cantilever is of 3m span. It carries a uniformly distributed load of 6 KN/m intensity on a 2m length of the span starting from the free end. Determine the slope & deflection at the

free end E = 205 GPa

Soln.



$\omega = 6 \text{ KN/m} = 6 \text{ N/mm}$

$$I = \frac{80 \times 180^3}{12} = 38.88 \times 10^6 \text{ mm}^4$$

Slope at free end

$$\frac{dy}{dx} = \frac{\omega L^3}{6EI} - \frac{\omega (L-a)^3}{6EI}$$

$$\frac{dy}{dx} = \frac{6}{6 \times 205 \times 1000 \times 38.88 \times 10^6}\left[3000^3 - (3000-2000)^3\right]$$

$$= 0.00326 \text{ rad}$$

Deflection at free end

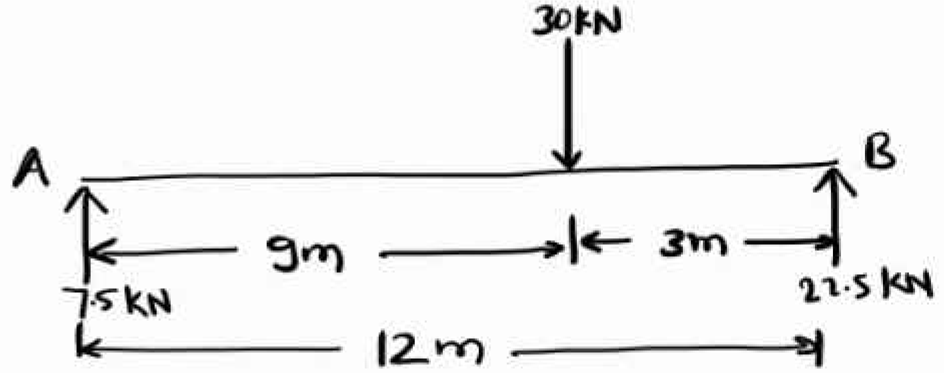
$$\text{Deflection at B} = \frac{\omega L^4}{8EI} - \left[\frac{\omega (L-a)^4}{8EI} + \frac{\omega (L-a)^3}{6EI} a\right]$$

$$= \frac{6}{2 \times 205 \times 10^3 \times 38.88 \times 10^6}\left[\frac{3000^4}{4} - \frac{(3000-2000)^4}{4} - \frac{(3000-2000)^3}{3} \times 2\right]$$

$$= 7.4 \text{mm}$$

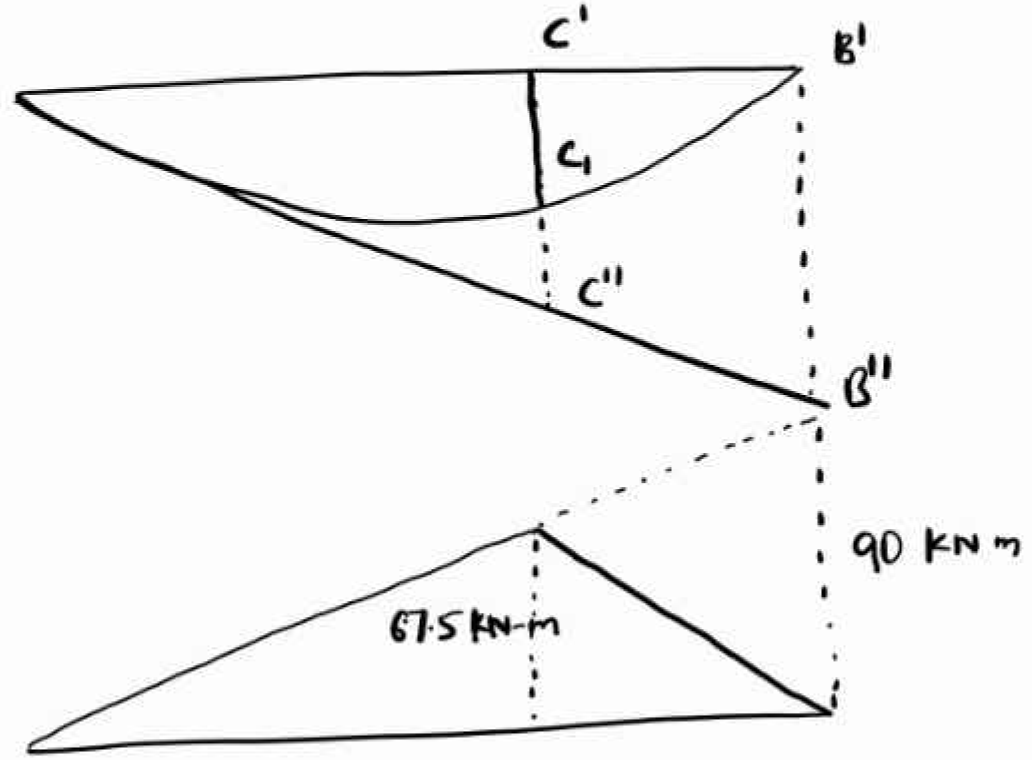
A simply supported beam of 12 m span carries a concentrated load of 30 kN at a distance of 9 m from the end A. Determine the deflection at the load point and slope at the load point. Take, $I = 2 \times 10^9\ mm^4$ + $E = 205\ GPa$

Sol^n.



$R_A = \frac{30 \times 3}{12} = 7.5\ kN$

$R_B = 30 - 7.5 = 22.5\ kN$



Applying Mohr's second moment area theorem

$$\frac{C'C''}{9} = \frac{B'B''}{12}$$

$$C'C'' = B'B'' \times \frac{9}{12}$$

But $B'B''$ is the intercept made by tangents at A & B on the elastic curve on a vertical line at B.

According to Mohr's second moment-area theorem, it must be equal to net moment of area of bending moment diagram between A & B about B divided by EI.

$$B'B'' = \frac{1}{EI}\left(\frac{1}{2}\times 90\times 12\times \frac{12}{3} - \frac{1}{2}\times 90\times 3\times \frac{3}{3}\right)$$

$$= \frac{1}{410\times 10^{12}}\left(\frac{1}{2}\times 90\times 12\times \frac{12}{3} - \frac{1}{2}\times 90\times 3\times \frac{3}{3}\right)\times 10^{12}$$

$$B'B'' = \left(\frac{45\times 4\times 12 - 45\times 3}{410}\right) = 4.93\text{mm}$$

$$C'C'' = B'B''\times\frac{9}{12} = 4.939\times\frac{9}{12}$$

$$C'C'' = 3.704 \text{ mm}$$

$C''C_1$ is the intercept made by tangents at C and A on the elastic curve on a vertical line at C.

According to Mohr's Second moment-area theorem

$$C_1C'' = \frac{\cancel{10^{12}}}{410\times\cancel{10^{12}}}\left(\frac{1}{2}\times 17.5\times \overset{3}{9}\times \overset{4.5}{9/\cancel{3}}\right)$$

$$C_1C'' = 2.22 \text{ mm}$$

Deflection at the load point.

$\therefore$ Deflection at C $= C'C'' - C_1C''$

$= 3.704 - 2.22$

Deflection at C $= 1.484$ mm

Slope at the load point

Slope at A $= \dfrac{B'B''}{l} = \dfrac{4.939}{12000} = 0.412 \times 10^{-3}$ rad

Slope at B w.r.t. A = Area of B.M. diagram between A & B

$$= \frac{10^9}{410 \times 10^{12}}\left(\frac{1}{2} \times 675 \times 9 + \frac{1}{2} \times 67.9 \times 3\right)$$

$$= \frac{1}{820 \times 10^3}\left(67.5 \times 9 + 67.9 \times 3\right)$$

$$= 0.989 \times 10^{-3} \text{ rad}$$

So, slope at C w.r.t. A = Area of BM diagram between A & C

$\therefore$ Slope at C $= 10^{-3}(0.989 - 0.412)$

$= 0.577 \times 10^{-3}$ rad